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Calculation of the Electroelastic Green's Function of the Hexagonal Infinite Medium

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Abstract

The electroelastic 4×4 Green's function of a piezoelectric hexagonal (transversely isotropic) infinitely extended medium is calculated explicitly in closed *compact* form (eqs. (73) ff. and (88) ff., respectively) by using residue calculation. The results can also be derived from Fredholm's method [2]. In the case of *vanishing* piezoelectric coupling the derived Green's function coincides with two well known results: Kröner's expressions for the elastic Green's function tensor [4] is reproduced and the electric part then coincides with the electric potential (solution of Poisson equation) which is caused by a unit point charge.

The obtained electroelastic Green's function is useful for the calculation of the electroelastic Eshelby tensor [16].

Keywords. Electroelastic Green's function, hexagonal medium, residues, piezoelectric materials

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1 Introduction

In the solution of many problems in physics Green's functions or fundamental solutions play an essential role. The elastic Green's function tensor has been derived for isotropic media by Sir Thompson [1] and implicitly for anisotropic media by Fredholm [2].

Especially the elastic Green's function of the infinite hexagonal medium has been calculated explicitly by Lifshitz and Rosenzweig [3] and by Kröner [4].

A first treatment of the electroelastic Green's function has been given by Deeg [5]. Further contributions have been presented by Dunn [6], Wang [7] and Huang and Yu [8] in the framework of inclusion problems. But no compact explicit form for the electroelastic Green's function has been given there. Thus, according to the author's knowledge, a *compact* closed form representation of the electroelastic Green's function does not exist. But, due to the widespread interest in piezoelectric materials a compact explicit representation is highly desirable.

The residue method which is applied here is a useful tool to obtain Green's functions in media with hexagonal symmetry as shown by Michelitsch [11] and Michelitsch and Wunderlin [12] for a treatment of the incompatibility problem.

In the following we calculate using this residue method the electroelastic 4×4 Green's function for the hexagonal infinite piezoelectric medium. We shall obtain this Green's function analytically in closed form (eqs. (73) ff. and (88) ff., respectively). To obtain a compact formulation the use of a convenient symmetric tensor basis is essential [9, 10]. The basis tensors used here are very useful for the solution of several problems in hexagonal media [11, 13, 14].

The result for the electroelastic Green's function presented here may be useful for the treatment of many problems, e.g. the inclusion problem in piezoelectric hexagonal (transversely isotropic) media.

2 Basic Equations

We start from the field equations for the stress tensor σ and the dielectric displacements D . The equilibrium conditions for the stresses are:

$$\partial_j \sigma_{ij} = -K_i, \quad \sigma_{ij} = \sigma_{ji} \quad (1)$$

∂_j indicate the spacial derivatives and \mathbf{K} the density of body forces, respectively. A further field equation describes the conservation of free electric charges:

$$\partial_l D_l = \rho_e \quad (2)$$

ρ_e represents the density of free electric charges. The constitutive equations (material law) of the piezoelectric medium which connect the electric field \mathbf{E} and the elastic deformation $\boldsymbol{\epsilon}$ with the dielectric displacement \mathbf{D} and the stress $\boldsymbol{\sigma}$ are given by:

$$\begin{aligned} \sigma_{ij} &= \mathcal{C}_{ijkl} \epsilon_{kl} - e_{kij} E_k \\ D_i &= e_{ikl} \epsilon_{kl} + \eta_{ik} E_k \end{aligned} \quad (3)$$

\mathcal{C}_{ijkl} , e_{kij} and η_{ij} denote the elastic moduli, the piezoelectric moduli and the dielectric moduli, respectively. They have the symmetry properties $\mathcal{C}_{ijkl} = \mathcal{C}_{klij} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk}$, $e_{ikl} = e_{ilk}$ and $\eta_{ij} = \eta_{ji}$. Introducing the electric potential Φ and the elastic displacement field \mathbf{u} the ansatz for the strain $\boldsymbol{\epsilon}$ and the electric field \mathbf{E} is given by:

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2} (\partial_i u_j + \partial_j u_i) \\ E_i &= -\partial_i \Phi \end{aligned} \quad (4)$$

Putting ansatz (4) using (3) into the field equations (1) and (2) we obtain a 4×4 differential equation of degree two for the field $\mathbf{U} = (\mathbf{u}, \Phi)$ of the form:

$$\mathcal{T}(\nabla) \mathbf{U} + \mathcal{F} = 0 \quad (5)$$

∇ indicates the gradient operator. Here we have introduced the generalized force density $\mathcal{F} = (\mathbf{K}, -\rho_e)$. The symmetric 4×4 second order differential operator $\mathcal{T}(\nabla)$ can be written in the form:

$$\mathcal{T}(\nabla) = \begin{bmatrix} \mathbf{T}(\nabla) & \mathbf{t}(\nabla) \\ \mathbf{t}^T(\nabla) & \tau(\nabla) \end{bmatrix} \quad (6)$$

Here $\mathbf{T}(\nabla)$ is a 3×3 tensor operator and represents the elastic part and is given by:

$$T_{ij}(\nabla) = \mathcal{C}_{ipjq} \partial_p \partial_q \quad (7)$$

$\mathbf{t}(\nabla)$ is a (3×1) tensor) vector operator given by

$$t_i(\nabla) = e_{piq} \partial_p \partial_q \quad (8)$$

and represents the piezoelectric coupling.

Finally the (1×1) tensor) scalar operator

$$\tau(\nabla) = -\eta_{pq} \partial_p \partial_q \quad (9)$$

describes the dielectric part.

The vector field \mathbf{u} can then be represented by the 4×4 electroelastic Green's function \mathcal{G} according to

$$\mathbf{u}(\mathbf{r}) = \int \mathcal{G}(\mathbf{r} - \mathbf{r}') \mathcal{F}(\mathbf{r}') d^3 \mathbf{r}' \quad (10)$$

\mathbf{r} denotes the space point. The electroelastic Green's function \mathcal{G} then is defined according to

$$\mathcal{T}(\nabla) \mathcal{G}(\mathbf{r}) + \delta^3(\mathbf{r}) \mathbf{1} = 0 \quad (11)$$

$\delta^3(\mathbf{r})$ represents the three-dimensional δ -function and $\mathbf{1}$ denotes the 4×4 unit matrix.

The electroelastic Green's function \mathcal{G}_{pq} ($p, q = 1, 2, 3, 4$) has the following physical interpretation [6]:

$\mathcal{G}_{mj}(\mathbf{r})$ ($m, j = 1, 2, 3$) is the elastic displacement at spacepoint \mathbf{r} in the m -direction caused by a unit point force at spacepoint $\mathbf{r}' = 0$ in the j -direction;

$\mathcal{G}_{m4}(\mathbf{r})$ ($m = 1, 2, 3$) is the elastic displacement at spacepoint \mathbf{r} in m -direction caused by a unit point charge at spacepoint $\mathbf{r}' = 0$;

$\mathcal{G}_{4j}(\mathbf{r})$ ($j = 1, 2, 3$) is the electric potential at spacepoint \mathbf{r} caused by a unit point force at spacepoint $\mathbf{r}' = 0$ in the j -direction;

$\mathcal{G}_{44}(\mathbf{r})$ is the electric potential at spacepoint \mathbf{r} caused by a unit point charge at spacepoint $\mathbf{r}' = 0$.

We note that our electroelastic Green's function defined in equation (11) is symmetric, i. e. $\mathcal{G}_{pq} = \mathcal{G}_{qp}$. This property follows from the symmetry of

the operator $\mathcal{T}_{rs}(\nabla) = \mathcal{T}_{sr}(\nabla)$ from equation (6). For our further calculation we use the following convenient representation for the Green's function [5, 9, 10, 11, 12]:

$$\mathcal{G}(\mathbf{r}) = \frac{1}{8\pi^2 r} \int_0^{2\pi} \mathcal{T}^{-1}(\boldsymbol{\xi}(\alpha)) d\alpha \quad (12)$$

This relation can be derived from (11) in straight forward manner by using Fourier transformation [11, 12]. One gets $\mathcal{T}(\boldsymbol{\xi})$ from equation (6) by replacing ∂_i by ξ_i ($i = 1, 2, 3$) in equations (7), (8) and (9). The vector $\boldsymbol{\xi}$ is then given by [11, 12]

$$\boldsymbol{\xi}(\alpha) = \mathbf{e}_1 \cos \alpha + \mathbf{e}_2 \sin \alpha \quad (13)$$

The vectors \mathbf{e}_i form a useful orthonormal basis

$$\mathbf{e}_1 = \frac{1}{\rho} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \frac{1}{\rho r} \begin{pmatrix} -zx \\ -zy \\ \rho^2 \end{pmatrix}, \quad \mathbf{e}_3 = \frac{1}{r} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (14)$$

where $\rho^2 = x^2 + y^2$, $r^2 = \rho^2 + z^2$ and $\mathbf{r} = r\mathbf{e}_3$. The orientation of this coordinate system can be expressed by $\mathbf{e}_i = \frac{1}{2}\epsilon_{ijk}\mathbf{e}_j \times \mathbf{e}_k$. ϵ_{ijk} denotes the antisymmetric permutation tensor.

3 Residue Calculation

Our goal is to formulate the residue calculation ansatz to obtain the Green's function from equation (12).

First of all we introduce for convenience a useful orthonormal basis to represent the 4×4 matrix $\mathcal{T}(\boldsymbol{\xi})$ and later $\mathcal{T}^{-1}(\boldsymbol{\xi})$:

$$\mathbf{e}_b = \frac{1}{\xi_b} \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_{b\perp} = \frac{1}{\xi_b} \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (15)$$

($\xi_b = \sqrt{\xi_1^2 + \xi_2^2}$, $\xi_c = \xi_3$). This basis represents the hexagonal symmetry. \mathbf{e}_b and $\mathbf{e}_{b\perp}$ are parallel to the basal plane and \mathbf{e}_c represents the c -direction. \mathbf{e}_4 comes into play because of the electric potential component Φ . We then obtain the following useful representation:

$$\begin{aligned} \mathcal{T}(\boldsymbol{\xi}) = & T_{b\perp} \mathbf{e}_{b\perp} \otimes \mathbf{e}_{b\perp} + T_b \mathbf{e}_b \otimes \mathbf{e}_b + T_{bc} (\mathbf{e}_b \otimes \mathbf{e}_c + \mathbf{e}_c \otimes \mathbf{e}_b) + T_c \mathbf{e}_c \otimes \mathbf{e}_c \\ & + t_{b4} (\mathbf{e}_b \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_b) + t_{c4} (\mathbf{e}_c \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_c) + \tau \mathbf{e}_4 \otimes \mathbf{e}_4 \end{aligned} \quad (16)$$

The scalar quantities $T_{b\perp}, T_b, T_{bc}, T_c, t_{b4}, t_{c4}$, and τ correspond to the tensors \mathbf{T}, \mathbf{t} and $\boldsymbol{\tau}$ from equations (7), (8) and (9), respectively (\otimes indicates dyadic multiplication). They are obtained as:

$$T_{b\perp} = \mathcal{C}_{66} \xi_b^2 + \mathcal{C}_{44} \xi_c^2, \quad (17)$$

$$T_b = \mathcal{C}_{11} \xi_b^2 + \mathcal{C}_{44} \xi_c^2, \quad (18)$$

$$T_{bc} = (\mathcal{C}_{13} + \mathcal{C}_{44}) \xi_b \xi_c, \quad (19)$$

$$T_c = \mathcal{C}_{44} \xi_b^2 + \mathcal{C}_{33} \xi_c^2, \quad (20)$$

$$t_{b4} = (e_{31} + e_{15}) \xi_b \xi_c, \quad (21)$$

$$t_{c4} = e_{15} \xi_b^2 + e_{33} \xi_c^2, \quad (22)$$

$$\tau = - \left(\eta_{11} \xi_b^2 + \eta_{33} \xi_c^2 \right) \quad (23)$$

$\mathcal{C}_{AB} = \{\mathcal{C}_{11}, \mathcal{C}_{44}, \mathcal{C}_{66}, \mathcal{C}_{13}, \mathcal{C}_{33}\}$ denote the elastic, $e_{iA} = \{e_{15}, e_{31}, e_{33}\}$ the piezoelectric, and $\eta_{ij} = \{\eta_{11}, \eta_{33}\}$ the dielectric moduli of the hexagonal material. Subscripts A, B represent Voigt's notation whereas i, j represent cartesian subscripts, respectively.

$\mathcal{T}^{-1}(\boldsymbol{\xi})$ can be written as:

$$\mathcal{T}^{-1}(\boldsymbol{\xi}) = \frac{\boldsymbol{\Lambda}(\boldsymbol{\xi})}{f(\boldsymbol{\xi})} \quad (24)$$

Here Λ_{ij} denotes the matrix of 3×3 subdeterminants (multiplied by the prefactor $(-1)^{i+j}$) and $f(\boldsymbol{\xi})$ the determinant of \mathcal{T} , respectively. We now can write for $\boldsymbol{\Lambda}$ by using (16):

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{\xi}) = & \Lambda_{b\perp} \mathbf{e}_{b\perp} \otimes \mathbf{e}_{b\perp} + \Lambda_b \mathbf{e}_b \otimes \mathbf{e}_b + \Lambda_{bc} (\mathbf{e}_b \otimes \mathbf{e}_c + \mathbf{e}_c \otimes \mathbf{e}_b) + \Lambda_c \mathbf{e}_c \otimes \mathbf{e}_c \\ & + \Lambda_{b4} (\mathbf{e}_b \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_b) + \Lambda_{c4} (\mathbf{e}_c \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_c) + \Lambda_4 \mathbf{e}_4 \otimes \mathbf{e}_4 \end{aligned} \quad (25)$$

Here the scalar quantities are introduced:

$$\Lambda_{b\perp} = \tau (T_b T_c - T_{bc}^2) - (t_{c4}^2 T_b - 2t_{c4} t_{b4} T_{bc} + t_{b4}^2 T_c), \quad (26)$$

$$\Lambda_b = T_{b\perp} (T_c \tau - t_{c4}^2), \quad (27)$$

$$\Lambda_{bc} = -T_{b\perp} (T_{bc} \tau - t_{b4} t_{c4}), \quad (28)$$

$$\Lambda_c = T_{b\perp} (T_b \tau - t_{b4}^2), \quad (29)$$

$$\Lambda_{b4} = T_{b\perp} (T_{bc} t_{c4} - T_c t_{b4}), \quad (30)$$

$$\Lambda_{c4} = -T_{b\perp} (T_b t_{c4} - T_{bc} t_{b4}), \quad (31)$$

$$\Lambda_4 = T_{b\perp} (T_b T_c - T_{bc}^2) \quad (32)$$

The determinant $f(\boldsymbol{\xi})$ of \mathcal{T} then takes the form

$$f(\boldsymbol{\xi}) = \text{Det} \mathcal{T}(\boldsymbol{\xi}) = T(\boldsymbol{\xi})_{b\perp} \Lambda_{b\perp}(\boldsymbol{\xi}) \quad (33)$$

Putting equations (17)-(23) into (33) shows that only terms proportional $\xi_b^{2n} \xi_c^{8-2n}$ ($n = 0, 1, 2, 3, 4$) appear. This is a unique property of the hexagonal medium. As we shall show this is the central property which is needed to solve our integration problem (12) explicitly. Thus (26) is a polynomial of degree 3 in a when we put $a = \xi_b^2 / \xi_c^2$. We then can write

$$\Lambda_{b\perp} = P(a) \xi_c^6 \quad (34)$$

where $P(a)$ is a polynomial of degree 3 in a and takes the form:

$$P(a) = Aa^3 + Ba^2 + Ca + D \quad (35)$$

We obtain for the coefficients A, B, C, D :

$$A = -\eta_{11}\mathcal{C}_{11}\mathcal{C}_{44} - \mathcal{C}_{11}e_{15}^2 \quad (36)$$

$$\begin{aligned} B = & -\eta_{33}\mathcal{C}_{11}\mathcal{C}_{44} - \eta_{11}(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2) - \mathcal{C}_{44}e_{15}^2 - 2\mathcal{C}_{11}e_{15}e_{33} \\ & + 2(\mathcal{C}_{13} + \mathcal{C}_{44})e_{15}(e_{31} + e_{15}) - \mathcal{C}_{44}(e_{31} + e_{15})^2 \end{aligned} \quad (37)$$

$$\begin{aligned} C = & -\eta_{33}(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2) - \eta_{11}\mathcal{C}_{33}\mathcal{C}_{44} - 2e_{15}e_{33}\mathcal{C}_{44} - e_{33}^2\mathcal{C}_{11} \\ & + 2e_{33}(e_{31} + e_{15})(\mathcal{C}_{13} + \mathcal{C}_{44}) - \mathcal{C}_{33}(e_{31} + e_{15})^2 \end{aligned} \quad (38)$$

$$D = -\eta_{33}\mathcal{C}_{33}\mathcal{C}_{44} - e_{33}^2\mathcal{C}_{44} \quad (39)$$

Thus we can factorize the determinant f according to:

$$f(\boldsymbol{\xi}) = \xi_c^8 \mathcal{C}_{66} A (a + A_1) (a + A_2) (a + A_3) (a + A_4), \quad (40)$$

with $T(a)_{b\perp} = \mathcal{C}_{66} (a + A_1) \xi_c^2$, $(A_1 = \mathcal{C}_{44}/\mathcal{C}_{66})$ and

$$P(a) = A (a + A_2) (a + A_3) (a + A_4) \quad (41)$$

The Λ 's from equations (26)-(32) yield with $a = \xi_b^2/\xi_c^2$ and $\xi_c^2 = 1$:

$$\Lambda_{b\perp}(a) = P(a) = Aa^3 + Ba^2 + Ca + D \quad (42)$$

The numbers A_2, A_3, A_4 are the zeros of the equation:

$$Aa^3 - Ba^2 + Ca - D = 0 \quad (43)$$

with above coefficients A, B, C, D from equations (36)-(39). The A_l are material quantities and fully determined by the moduli $\mathbf{C}, \mathbf{e}, \boldsymbol{\eta}$. Furthermore, the subdeterminants (27)-(32) yield:

$$\Lambda_b(a) = -(\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[(\eta_{11}a + \eta_{33})(\mathcal{C}_{44}a + \mathcal{C}_{33}) + (e_{15}a + e_{33})^2 \right], \quad (44)$$

$$\Lambda_{bc}(a) = \sqrt{a}(\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[(e_{31} + e_{15})(e_{15}a + e_{33}) + (\eta_{11}a + \eta_{33})(\mathcal{C}_{13} + \mathcal{C}_{44}) \right], \quad (45)$$

$$\Lambda_c(a) = -(\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[(\eta_{11}a + \eta_{33})(\mathcal{C}_{11}a + \mathcal{C}_{44}) + a(e_{31} + e_{15})^2 \right], \quad (46)$$

$$\Lambda_{b4}(a) = \sqrt{a}(\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[(\mathcal{C}_{13} + \mathcal{C}_{44})(e_{15}a + e_{33}) - (\mathcal{C}_{44}a + \mathcal{C}_{33})(e_{31} + e_{15}) \right], \quad (47)$$

$$\Lambda_{c4}(a) = -(\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[(\mathcal{C}_{11}a + \mathcal{C}_{44})(e_{15}a + e_{33}) - a(\mathcal{C}_{13} + \mathcal{C}_{44})(e_{31} + e_{15}) \right], \quad (48)$$

$$\Lambda_4(a) = (\mathcal{C}_{66}a + \mathcal{C}_{44}) \left[a^2\mathcal{C}_{11}\mathcal{C}_{44} + a(\mathcal{C}_{11}\mathcal{C}_{33} - 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{13}^2) + \mathcal{C}_{33}\mathcal{C}_{44} \right] \quad (49)$$

The dependence on a is obtained by putting $\xi_b = \sqrt{a}$ and $\xi_c = 1$ in equations (17)-(23) and (26)-(32), respectively.

To evaluate (12) we make use of the properties:

$$\Lambda_{ij}(\xi\lambda) = \lambda^6 \Lambda_{ij}(\xi) \quad (50)$$

λ denotes an arbitrary scalar number. The components Λ_{ij} are homogeneous functions of degree 6. Whereas the determinant f is homogeneous of degree 8:

$$f(\xi\lambda) = \lambda^8 f(\xi) \quad (51)$$

Thus

$$\mathcal{T}^{-1}(\xi) = \lambda^2 \frac{\Lambda(\lambda\xi)}{f(\lambda\xi)} \quad (52)$$

We now introduce the complex vector [12]

$$\boldsymbol{\gamma}(\alpha) = 2e^{i\alpha}\boldsymbol{\xi}(\alpha) \quad (53)$$

with $\boldsymbol{\xi}(\alpha)$ from equation (13) (i denotes the imaginary unit). (53) can then be written as

$$\boldsymbol{\gamma}(s) = \boldsymbol{h}^*s + \boldsymbol{h} \quad (54)$$

Here we have introduced the complex variable

$$s = e^{2i\alpha} \quad (55)$$

and the vector

$$\boldsymbol{h} = \boldsymbol{e}_1 + i\boldsymbol{e}_2 \quad (56)$$

with the basis vectors $\boldsymbol{e}_{1,2}$ from equations (14). When putting $\lambda = 2e^{i\alpha}$ equation (52) then can be written as:

$$\mathcal{T}^{-1}(\boldsymbol{\xi}) = 4s \frac{\boldsymbol{\Lambda}(\boldsymbol{\gamma}(s))}{f(\boldsymbol{\gamma}(s))} \quad (57)$$

Furthermore we observe

$$ds = 2isd\alpha \quad (58)$$

Then the integral (12) can be transformed into a complex integral over the unit circle:

$$\mathcal{G}(\boldsymbol{r}) = \frac{1}{8\pi^2 r} \oint_{|s|=1} \frac{4\boldsymbol{\Lambda}(\boldsymbol{\gamma}(s))}{f(\boldsymbol{\gamma}(s))} \frac{ds}{i} \quad (59)$$

Using the residue theorem we can rewrite integral (59) as

$$\mathcal{G}(\boldsymbol{r}) = \frac{1}{8\pi^2 r} 2\pi i \sum \text{Res} \left(\frac{4\boldsymbol{\Lambda}(\boldsymbol{\gamma}(s))}{if(\boldsymbol{\gamma}(s))} \right) \quad (60)$$

To evaluate (60) we have to find all zeros of $f(\boldsymbol{\gamma}(s))$ which are located within the unit circle. To find these zeros we observe that $f(\boldsymbol{\gamma}(s))$ is a polynomial of degree 8 in s . Thus there exist 8 zeros s_j of f . We assume here that f has no multiple zeros s_j .

Following [12] we can conclude that there are 4 pairs of zeros s_l, \bar{s}_l ($l = 1, 2, 3, 4$) having the property:

$$|s_l \bar{s}_l| = 1 \quad (61)$$

There are only four zeros s_l which are located within the unit circle. Four zeros \bar{s}_l lie outside the unit circle. Thus the zeros can be written in the form [12]:

$$s_l = e^{2i\Phi_l} e^{-2\psi_l}, \quad (62)$$

$$\bar{s}_l = e^{2i\Phi_l} e^{+2\psi_l}, \quad (63)$$

where $\psi_l > 0$ ($l = 1, 2, 3, 4$). Here the degenerate case $|s_l| = |\bar{s}_l|$ is excluded by our assumption of no multiple zeros. Only the residues corresponding to the four zeros (62) lying *within* the unit circle contribute to (60). The residues at $s = s_l$ are obtained as:

$$\text{Res} \left(\frac{4\Lambda(\gamma(s))}{if(\gamma(s))} \right) \Big|_{s=s_l} = \frac{4\Lambda(\gamma(s_l))}{i \frac{df(\gamma(s))}{ds} \Big|_{s=s_l}} \quad (64)$$

Finally we arrive at

$$\mathcal{G}(r) = \frac{1}{4\pi r} \sum_{l=1}^4 \frac{4\Lambda(\gamma(s_l))}{\frac{df(\gamma(s))}{ds} \Big|_{s=s_l}} \quad (65)$$

The zeros s_l of $f(\gamma(s))$ within the unit circle yield

$$s_l = \frac{\sqrt{A_l \rho^2 + z^2} - r}{\sqrt{A_l \rho^2 + z^2} + r} \quad (66)$$

To obtain (66) equation (40) together with (54) and (56) is used.

4 Explicit form of the Green's function

Evaluating (65) the Green's function takes the form:

$$\mathcal{G}(r) = \frac{1}{4\pi A \mathcal{C}_{66}} \sum_{l=1}^4 \frac{\Lambda(\xi^{(l)})}{\sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (67)$$

$\boldsymbol{\xi}^{(l)}$ ($l = 1, 2, 3, 4$) are the solutions of the equations

$$f(\boldsymbol{\xi}) = 0 \quad (68)$$

or

$$\xi_1^2 + \xi_2^2 + A_l \xi_3^2 = 0 \quad (69)$$

and

$$\boldsymbol{\xi} \mathbf{r} = \xi_1 x + \xi_2 y + \xi_3 z = 0 \quad (70)$$

with $\xi_3^{(l)} = 1$. Thus equations (68) and (70) have the solutions ($l = 1, 2, 3, 4$):

$$\xi_1^{(l)} = \frac{1}{\rho^2} \left[-zx + iy\sqrt{A_l \rho^2 + z^2} \right] \quad (71)$$

$$\xi_2^{(l)} = \frac{1}{\rho^2} \left[-zy - ix\sqrt{A_l \rho^2 + z^2} \right] \quad (72)$$

($\rho = \sqrt{x^2 + y^2}$). This formulation is in complete analogy to the result obtained by Kröner for the Green's function tensor of the hexagonal elastic medium [4] by using Fredholm's method [2].

The Green's function (67) then assumes the form:

$$\begin{aligned} \mathcal{G}(\mathbf{r}) = & \mathcal{G}_{\phi\phi}(\rho, z) \mathbf{e}_\phi \otimes \mathbf{e}_\phi + \mathcal{G}_{\rho\rho}(\rho, z) \mathbf{e}_\rho \otimes \mathbf{e}_\rho + \mathcal{G}_{\rho z}(\rho, z) (\mathbf{e}_\rho \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\rho) \\ & + \mathcal{G}_{zz}(\rho, z) \mathbf{e}_z \otimes \mathbf{e}_z + \mathcal{G}_{\rho 4}(\rho, z) (\mathbf{e}_\rho \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_\rho) \\ & + \mathcal{G}_{z4}(\rho, z) (\mathbf{e}_z \otimes \mathbf{e}_4 + \mathbf{e}_4 \otimes \mathbf{e}_z) + \mathcal{G}_{44}(\rho, z) \mathbf{e}_4 \otimes \mathbf{e}_4 \end{aligned} \quad (73)$$

Here we have introduced the following basis:

$$\mathbf{e}_\rho = \frac{1}{\rho} \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_\phi = \frac{1}{\rho} \begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (74)$$

To obtain (73) we have used (25) together with (71) and (72). Equation (67) then yields the \mathcal{G} 's introduced in (73) as:

$$\mathcal{G}_{\phi\phi}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{\rho^2 \Lambda_b(a = -A_l) + z^2 \Gamma_b(a = -A_l)}{\rho^2 \sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (75)$$

$$\mathcal{G}_{\rho\rho}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{\rho^2 \Lambda_{b\perp}(a = -A_l) - z^2 \Gamma_b(a = -A_l)}{\rho^2 \sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (76)$$

$$\mathcal{G}_{\rho z}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{(-z) \Gamma_{bc}(a = -A_l)}{\rho \sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (77)$$

$$\mathcal{G}_{\rho^4}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{(-z) \Gamma_{b4}(a = -A_l)}{\rho \sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (78)$$

$$\mathcal{G}_{zz}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{\Lambda_c(a = -A_l)}{\sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (79)$$

$$\mathcal{G}_{z^4}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{\Lambda_{c4}(a = -A_l)}{\sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (80)$$

$$\mathcal{G}_{44}(\rho, z) = \frac{1}{4\pi A\mathcal{C}_{66}} \sum_{l=1}^4 \frac{\Lambda_4(a = -A_l)}{\sqrt{A_l \rho^2 + z^2} \prod_{j=1, (j \neq l)}^4 (A_j - A_l)} \quad (81)$$

Here we have introduced the quantities

$$\Lambda_{b\perp}(a) - \Lambda_b(a) = a\Gamma_b(a) \quad (82)$$

to arrive at

$$\begin{aligned} \Gamma_b(a) &= (\mathcal{C}_{11} - \mathcal{C}_{66}) (T_c(a) \tau(a) - t_{c4}^2(a)) \\ &\quad - (\mathcal{C}_{13} + \mathcal{C}_{44})^2 \tau(a) + 2(\mathcal{C}_{13} + \mathcal{C}_{44}) (e_{31} + e_{15}) t_{c4}(a) \\ &\quad + (e_{31} + e_{15})^2 T_c(a) \end{aligned} \quad (83)$$

and

$$\Lambda_{bc}(a) = \sqrt{a} \Gamma_{bc}(a) \quad (84)$$

where (compare eq. (45))

$$\Gamma_{bc}(a) = (\mathcal{C}_{66}a + \mathcal{C}_{44}) [(e_{31} + e_{15})(e_{15}a + e_{33}) + (\eta_{11}a + \eta_{33})(\mathcal{C}_{13} + \mathcal{C}_{44})], \quad (85)$$

and

$$\Lambda_{b4}(a) = \sqrt{a}\Gamma_{b4}(a) \quad (86)$$

where (compare eq. (47))

$$\Gamma_{b4}(a) = (\mathcal{C}_{66}a + \mathcal{C}_{44}) [(\mathcal{C}_{13} + \mathcal{C}_{44})(e_{15}a + e_{33}) - (\mathcal{C}_{44}a + \mathcal{C}_{33})(e_{31} + e_{15})] \quad (87)$$

Equation (73) together with (75)-(81) represents the electroelastic Green's function explicitly in compact form. Especially for the calculation of the electroelastic Eshelby tensor the use of representation (73) may be convenient.

The cartesian representation of the the electroelastic Green's function becomes:

$$\mathcal{G}(\mathbf{r}) = \sum_{l=1}^4 \frac{1}{\sqrt{A_l \rho^2 + z^2}} \times \begin{pmatrix} g_{11}^{(l)} & g_{12}^{(l)} & g_{13}^{(l)} & g_{14}^{(l)} \\ g_{12}^{(l)} & g_{22}^{(l)} & g_{23}^{(l)} & g_{24}^{(l)} \\ g_{13}^{(l)} & g_{23}^{(l)} & g_{33}^{(l)} & g_{34}^{(l)} \\ g_{14}^{(l)} & g_{24}^{(l)} & g_{34}^{(l)} & g_{44}^{(l)} \end{pmatrix} \quad (88)$$

Here we have used the abbreviations:

$$g_{11}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_b(-A_l) \frac{x^2 z^2 - y^2 (A_l \rho^2 + z^2)}{\rho^4} + \Lambda_{b\perp}(-A_l) \right] \quad (89)$$

$$g_{22}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_b(-A_l) \frac{y^2 z^2 - x^2 (A_l \rho^2 + z^2)}{\rho^4} + \Lambda_{b\perp}(-A_l) \right] \quad (90)$$

$$g_{12}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_b(-A_l) \frac{xy (A_l \rho^2 + 2z^2)}{\rho^4} \right] \quad (91)$$

$$g_{13}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_{bc}(-A_l) \frac{xz}{\rho^2} \right] \quad (92)$$

$$g_{23}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_{bc} (-A_l) \frac{yz}{\rho^2} \right] \quad (93)$$

$$g_{33}^{(l)} = \frac{1}{\mathcal{E}_l} \Lambda_c (-A_l) \quad (94)$$

$$g_{14}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_{b4} (-A_l) \frac{xz}{\rho^2} \right] \quad (95)$$

$$g_{24}^{(l)} = \frac{1}{\mathcal{E}_l} \left[-\Gamma_{b4} (-A_l) \frac{yz}{\rho^2} \right] \quad (96)$$

$$g_{34}^{(l)} = \frac{1}{\mathcal{E}_l} \Lambda_{c4} (-A_l) \quad (97)$$

$$g_{44}^{(l)} = \frac{1}{\mathcal{E}_l} \Lambda_4 (-A_l) \quad (98)$$

and

$$\mathcal{E}_l = 4\pi\mathcal{C}_{66}A \prod_{j=1, (j \neq l)}^4 (A_j - A_l) \quad (99)$$

As we shall show in the appendix the Green's function (88) yields in the case of vanishing piezoelectric coupling $e_{ijk} = 0$ the well known results: For the elastic part \mathcal{G}_{ij} (i,j=1,2,3) Kröner's elastic Green's tensor of the hexagonal medium [4] and the dielectric part \mathcal{G}_{44} then represents the solution of Poisson equation of a unit point charge in a hexagonal dielectric medium, whereas the \mathcal{G}_{j4} ($j = 1, 2, 3$) are vanishing.

5 Appendix

Here we consider the case of vanishing piezoelectric coupling ($e_{ijk} = 0$). As a consequence the terms (eqs. (21), (22))

$$t_{b4} = t_{c4} = 0 \quad (100)$$

are vanishing. Thus the determinant f (eq. (33)) simplifies according to

$$f(a) = \text{Det}\mathcal{T}(a) = \tau(a) T_{b\perp}(a) \left(T_b(a) T_c(a) - T_{bc}^2(a) \right) \quad (101)$$

and can be written as

$$f(a) = -\eta_{11} \mathcal{C}_{11} \mathcal{C}_{44} \mathcal{C}_{66} (a + a_1) (a + a_2) (a + a_3) (a + a_4) \quad (102)$$

where $a_l = A_l$ denote the zeros of $f(-a)$ in the decoupled case.

$$a_1 = \frac{\mathcal{C}_{44}}{\mathcal{C}_{66}} \quad (103)$$

represents the zero of $T_{b\perp}(-a)$. The numbers $a_{2,3}$ are the zeros of the quadratic equation

$$T_b(-a) T_c(-a) - T_{bc}^2(-a) = \mathcal{C}_{11} \mathcal{C}_{44} a^2 + (\mathcal{C}_{13}^2 + 2\mathcal{C}_{13} \mathcal{C}_{44} - \mathcal{C}_{11} \mathcal{C}_{33}) a + \mathcal{C}_{33} \mathcal{C}_{44} = 0 \quad (104)$$

The zeros a_l for $l = 1, 2, 3$ (elastic part) are those introduced in [4]. a_4 is given by

$$a_4 = \frac{\eta_{33}}{\eta_{11}} \quad (105)$$

and represents the zero of $\tau(-a)$. Furthermore we find when using (42)-(49) together with (82)-(87)

$$\Lambda_{b\perp}(a = -a_l) = -\eta_{11} \mathcal{C}_{11} \mathcal{C}_{44} (a_2 - a_l) (a_3 - a_l) (a_4 - a_l) \quad (106)$$

$$\Lambda_b(a = -a_l) = -\eta_{11} \mathcal{C}_{66} (\mathcal{C}_{33} - a_l \mathcal{C}_{44}) (a_1 - a_l) (a_4 - a_l) \quad (107)$$

$$\Gamma_b(a = -a_l) = \eta_{11} (a_4 - a_l) \left[(\mathcal{C}_{66} - \mathcal{C}_{11}) (\mathcal{C}_{33} - a_l \mathcal{C}_{44}) + (\mathcal{C}_{13} + \mathcal{C}_{44})^2 \right] \quad (108)$$

$$\Gamma_{bc}(a = -a_l) = \eta_{11} \mathcal{C}_{66} (\mathcal{C}_{13} + \mathcal{C}_{44}) (a_1 - a_l) (a_4 - a_l) \quad (109)$$

$$\Lambda_c(a = -a_l) = -\eta_{11} \mathcal{C}_{66} (\mathcal{C}_{44} - a_l \mathcal{C}_{11}) (a_1 - a_l) (a_4 - a_l) \quad (110)$$

and

$$\Gamma_{b4}(a = -a_l) = \Lambda_{c4}(a = -a_l) = 0 \quad (111)$$

to obtain $\mathcal{G}_{j4} = 0$ ($j = 1, 2, 3$) and

$$\Lambda_4(a = -a_l) = \mathcal{C}_{11}\mathcal{C}_{44}\mathcal{C}_{66}(a_1 - a_l)(a_2 - a_l)(a_3 - a_l) \quad (112)$$

and for $l = 1, 2, 3$:

$$\mathcal{E}_l = -\eta_{11}(a_4 - a_l)E_l \quad (113)$$

with

$$E_l = 4\pi\mathcal{C}_{11}\mathcal{C}_{44}\mathcal{C}_{66} \prod_{j=1, (j \neq l)}^3 (a_j - a_l) \quad (114)$$

The terms (114) coincide with Kröner's (corrected) " E_l " (the terms E_l defined in [4] have to be corrected by a prefactor $-\mathcal{C}_{11}\mathcal{C}_{44}\mathcal{C}_{66}$ to obtain the correct result for the elastic Green's tensor [15]).

For $l = 4$ we obtain:

$$\mathcal{E}_4 = -4\pi\eta_{11}\mathcal{C}_{11}\mathcal{C}_{44}\mathcal{C}_{66}(a_1 - a_4)(a_2 - a_4)(a_3 - a_4) = -4\pi\eta_{11}\Lambda_4(-a_4) \quad (115)$$

We observe from equation (106)-(110) the properties:

$$\frac{\Lambda_{b\perp}(a = -a_4)}{\mathcal{E}_4} = \frac{\Lambda_b(a = -a_4)}{\mathcal{E}_4} = \frac{\Gamma_b(a = -a_4)}{\mathcal{E}_4} = \frac{\Gamma_{bc}(a = -a_4)}{\mathcal{E}_4} = \frac{\Lambda_c(a = -a_4)}{\mathcal{E}_4} = 0 \quad (116)$$

($\mathcal{E}_4 \neq 0$, eq. (115)). Thus the term $l = 4$ in sum (88) does not contribute to the elastic components \mathcal{G}_{ij} ($i, j = 1, 2, 3$). Using equations (112) and (113) together with (114) we obtain for $l = 1, 2, 3$:

$$\frac{\Lambda_4(a = -a_l)}{\mathcal{E}_l} = 0 \quad (117)$$

Thus there are no contributions to sum (88) for $l = 1, 2, 3$ to the dielectric part \mathcal{G}_{44} . For $l = 4$ we find from equations (112) and (115):

$$\frac{\Lambda_4(a = -a_4)}{\mathcal{E}_4} = \frac{-1}{4\pi\eta_{11}} \quad (118)$$

(118) is independent on the elastic moduli \mathcal{C}_{ijkl} which is a consequence of $e_{ijk} = 0$. Thus the dielectric part \mathcal{G}_{44} of the Green's function (88) yields

$$\mathcal{G}_{44}(\mathbf{r}) = \frac{\Lambda_4(a = -a_4)}{\mathcal{E}_4} \frac{1}{\sqrt{a_4\rho^2 + z^2}} = \frac{-1}{4\pi\eta_{11}\sqrt{a_4\rho^2 + z^2}} \quad (119)$$

where $a_4 = \eta_{33}/\eta_{11}$. Indeed it is easily checked that (119) is the solution of the Poisson equation of a unit point charge (compare (9) and (11))

$$\tau(\nabla)\mathcal{G}_{44}(\mathbf{r}) + \delta^3(\mathbf{r}) = 0 \quad (120)$$

where (compare equation (9))

$$\tau(\nabla) = -\left[\eta_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \eta_{33}\frac{\partial^2}{\partial z^2}\right] \quad (121)$$

Let us consider now the cases $l = 1, 2, 3$:

From equations (106)-(110) together with (113) we obtain:

$$-\frac{\Gamma_b(a = -a_l)}{\mathcal{E}_l} = \mathcal{A}_l = \frac{(\mathcal{C}_{66} - \mathcal{C}_{11})(\mathcal{C}_{33} - a_l\mathcal{C}_{44}) + (\mathcal{C}_{13} + \mathcal{C}_{44})^2}{E_l} \quad (122)$$

These terms coincide with Kröner's " \mathcal{A}_l " from [4]. Furthermore we obtain:

$$\frac{\Lambda_{b\perp}(a = -a_l)}{\mathcal{E}_l} = B_l = \frac{\mathcal{C}_{11}\mathcal{C}_{44}a_l^2 + (\mathcal{C}_{13}^2 + 2\mathcal{C}_{13}\mathcal{C}_{44} - \mathcal{C}_{11}\mathcal{C}_{33})a_l + \mathcal{C}_{33}\mathcal{C}_{44}}{E_l} \quad (123)$$

These terms coincide with Kröner's " B_l " from [4].

Furthermore we obtain

$$-\frac{\Gamma_{bc}(a = -a_l)}{\mathcal{E}_l} = C_l = \frac{(\mathcal{C}_{44} - a_l\mathcal{C}_{66})(\mathcal{C}_{13} + \mathcal{C}_{44})}{E_l} \quad (124)$$

These terms coincide with Kröner's " C_l " from [4].

Finally we obtain

$$\frac{\Lambda_c (a = -a_l)}{\mathcal{E}_l} = D_l = \frac{(\mathcal{C}_{44} - a_l \mathcal{C}_{66})(\mathcal{C}_{44} - a_l \mathcal{C}_{11})}{E_l} \quad (125)$$

These terms coincide with Kröner's "D_l" from [4].

Because of the property (116) the elastic part of the Green's function $\mathcal{G}_{ij} = G_{ij}$ ($i, j = 1, 2, 3$) yields by putting (122)-(125) into (88) and by using the abbreviations (89)-(94) Kröner's result [4]:

$$\mathbf{G}(\mathbf{r}) = \sum_{l=1}^3 \frac{1}{\sqrt{a_l \rho^2 + z^2}} \times \begin{pmatrix} \mathcal{A}_l \frac{x^2 z^2 - y^2 (a_l \rho^2 + z^2)}{\rho^4} + B_l; & \mathcal{A}_l \frac{xy (a_l \rho^2 + 2z^2)}{\rho^4}; & C_l \frac{xz}{\rho^2} \\ \mathcal{A}_l \frac{xy (a_l \rho^2 + 2z^2)}{\rho^4}; & \mathcal{A}_l \frac{y^2 z^2 - x^2 (a_l \rho^2 + z^2)}{\rho^4} + B_l; & C_l \frac{yz}{\rho^2} \\ C_l \frac{xz}{\rho^2}; & C_l \frac{yz}{\rho^2}; & D_l \end{pmatrix} \quad (126)$$

Here we used Kröner's notation. Above constants $\mathcal{A}_l, B_l, C_l, D_l$ are defined in equations (122)-(125), respectively.

6 Conclusion

The electroelastic 4×4 Green's function of a piezoelectric hexagonal medium which is infinitely extended has been calculated explicitly by using residue theory. The obtained expression (88) is highly convenient for a treatment of the electroelastic Eshelby tensor [16].

An important future application may be the following: The obtained Green's function will be an important quantity even for a treatment of nonlinear, e.g. hysteretic material behavior in piezoelectric ceramics. To model these effects has become of great interest since the increasing technological importance of piezoelectric materials, in particular as electromechanical actuators and sensors [17, 18, 19]. Key contributions for a treatment of such nonlinear effects

using the technique of Green's functions had been presented by Wunderlin and Haken [20, 21].

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